数理解析研究所講究録 1428 巻 2005 年 1-8

Perron's method – revisited –

Shigeaki Koike (小池 茂昭)
Saitama University (埼玉大学)

1 Introduction

According to [7] (p. 24), the classical Perron's method asserts that

$$u(x) := \sup_{v\in S_\phi} v(x) \text{ is harmonic,}$$

where S_ϕ denotes the set of subharmonic functions $v \in C^2(\Omega)\cap C(\overline{\Omega})$ such that $v \leq \phi$ on $\partial\Omega$. Here $\phi \in C(\partial\Omega)$ is a given function and $\Omega \subset \mathbf{R}^n$ an open bounded set. See also the original paper [16].

Motivated by this classical Perron's method, H. Ishii in [8] established a celebrated existence result for viscosity solutions. Although it only gives the existence of possibly discontinuous viscosity solutions, applying the comparison principle (under suitable boundary conditions and/or growth conditions when Ω is unbounded), we immediately obtain the continuity of these viscosity solutions.

However, there still exist some types of (uniformly) elliptic PDEs for which we do not know if the comparison principle holds true for viscosity solutions. A typical situation is the case when the coefficients of the PDE are only measurable. In fact, in general, under such a condition, it is known that there is a counter-example for the uniqueness of viscosity solutions in [15] (see also [17]). This indicates that the comparison principle does not hold since it implies the uniqueness of viscosity solutions.

We will discuss the other type of PDEs for which we do not know if the comparison principle for viscosity solutions holds.

In this abstract, we first present a modified Perron's method for possibly degenerate elliptic PDEs.

Next, in the case when PDEs are uniformly elliptic, we show that our viscosity solutions constructed by our Perron's method has local Hölder continuity estimate. We note that since we do not know that our viscosity solutions are continuous a priori, we cannot apply the argument by H. Ishii and P.-L. Lions in [9] to show the Hölder estimate. Thus, we will follow Caffarelli's argument in [2] (see also [3]).

2 Preliminaries

We are concerned with fully nonlinear second-order elliptic partial differential equations (PDEs for short):

$$G(x, Du, D^2u) = f(x) \quad \text{in } \Omega, \tag{1}$$

where $G : \Omega \times \mathbf{R}^n \times S^n \to \mathbf{R}$ and $f : \Omega \to \mathbf{R}$ are given. Although we may consider the case when G has u-variable, for the sake of simplicity, we restrict ourselves to study the PDE (1).

In what follows, we suppose that $\Omega \subset \mathbf{R}^n$ is only an open (possibly unbounded) set.

We will use the following notation: For $r > 0$ and $x \in \mathbf{R}^n$,

$$B_r(x) = \{y \in \mathbf{R}^n \mid |x - y| < r\} \text{ and } Q_r(x) = \{y \in \mathbf{R}^n \mid \max_{i=1,\ldots,n} |x_i - y_i| < r/2\}.$$

We will simply write B_r and Q_r, respectively, for $B_r(0)$ and $Q_r(0)$.

We suppose that

$$f \in L^p_{loc}(\Omega) \quad \text{for } p \geq p^*, \tag{2}$$

where $p^* \in (n/2, n)$ depends only on n and Λ/λ. Here, the uniform ellipticity constans $0 < \lambda \leq \Lambda$ will be fixed in section 4. We refer to [6] for the dependence of p^*.

Definition. We call $u : \Omega \to \mathbf{R}$ an L^p-viscosity subsolution (resp., supersolution) of (1) if the following property is satisfied: For any $\phi \in W^{2,p}_{loc}(\Omega)$ and for any local maximum (resp., minimum) point $x \in \Omega$ of $u^* - \phi$ (resp., $u_* - \phi$), we have

$$\lim_{\varepsilon \to 0} ess. \inf_{B_\varepsilon(x)} \{G(y, D\phi(y), D^2\phi(y)) - f(y)\} \leq 0$$

$$\left(\text{resp., } \lim_{\varepsilon \to 0} ess. \sup_{B_\varepsilon(x)} \{G(y, D\phi(y), D^2\phi(y)) - f(y)\} \geq 0\right).$$

We also call $u : \Omega \to \mathbf{R}$ an L^p-viscosity solution of (1) if it is both an L^p-viscosity sub- and supersolution of (1).

Remarks. (1) We denote by u^* and u_*, respectively, the upper and lower semicontinuous envelopes of u. We refer to [5] for the definitions.

(2) We also note that $W^{2,p}_{loc}(\Omega) \subset C(\Omega)$ for $p > n/2$.

We denote the set of modulus of continuity by

$$\mathcal{M} = \{\omega \in C([0, \infty)) \mid \omega(0) = 0\}.$$

In addition to (2), we will suppose the following:

$$\begin{cases} \text{For any compact set } K \subset \Omega, \text{ there is } \omega_K \in \mathcal{M} \text{ such that} \\ |G(x,q,X) - G(x,q',X')| \le \omega_K(|q-q'| + |X-X'|) \\ \text{for } x \in K, q, q' \in \mathbf{R}^n, X, X' \in S^n. \end{cases} \tag{3}$$

In order to deal with PDEs with quadratic nonlinearity, $|Du|^2$, we will use the following continuity assumption for G under (2) with $p > n$:

$$\begin{cases} \text{For any compact set } K \subset \Omega \text{ and } R > 0, \text{ there is } \omega_{K,R} \in \mathcal{M} \\ \text{such that } |G(x,q,X) - G(x,q',X')| \le \omega_{K,R}(|q-q'| + |X-X'|) \\ \text{for } x \in K, q, q' \in B_R, X, X' \in S^n. \end{cases} \tag{4}$$

We shall verify that under certain condition on G we may suppose the "strict" maximum (reps., minimum) in our definition of L^p-viscosity solutions. In fact, in the proof of Perron's method, we need to replace the maximum (resp., minimum) point in the definition by the strict one.

Proposition 1. *Assume one of the following properties:*

$$\begin{cases} (i) & (2) \text{ and } (3) \text{ hold.} \\ (ii) & (2) \text{ with } p > n \text{ and } (4) \text{ hold.} \end{cases} \tag{5}$$

Then, we can replace the "maximum (resp., minimum)" in the definition of L^p-viscosity subsolutions (resp., supersolution) by the "strict maximum (resp., minimum).

Remark. The reason why we suppose $p > n$ under (4) is that we need to know the local bound of the gradient of $\phi \in W^{2,p}_{loc}(\Omega)$ in the definition.

3 Modified Perron's method

We shall first give a sort of stability results.

Proposition 2. *Assume that (5) holds. Let $\mathcal{S} \subset C(\Omega)$ be a non-empty set of L^p-viscosity subsolutions (resp., supersolution) of (1). Assume also that $u(x) := \sup_{v \in \mathcal{S}} v(x)$ (resp., $:= \inf_{v \in \mathcal{S}} v(x)$) is locally bounded in Ω.*

Then, u is an L^p-viscosity subsolution (resp., supersolution) of (1).

In this section, we suppose that G is degenerate elliptic;

$$\begin{cases} G(x,q,X) \le G(x,q,Y) \\ \text{for } x \in \Omega, q \in \mathbf{R}^n, X, Y \in S^n \text{ with } X \ge Y. \end{cases} \tag{6}$$

We next show that "strong" solutions in $W^{2,p}_{loc}(\Omega)$ are indeed L^p-viscosity solutions. This fact is necessary to prove our Perron's method since we deal with L^p-viscosity solutions.

Proposition 3. (cf. [4], [14]) *Assume that* (5) *and* (6) *hold. If* $u \in W^{2,p}_{loc}(\Omega)$ *satisfies*

$$G(x, Du(x), D^2u(x)) \leq f(x) \quad (\text{resp.}, \geq 0) \quad \text{a.e. in } \Omega,$$

then u *is an* L^p*-viscosity subsolution (resp., supersolution) of* (1).

Our existence result via Perron's method is as follows:

Theorem 4. *Assume that* (5) *and* (6) *hold. Assume also that there are an* L^p*-viscosity subsolution* $\underline{u} \in C(\Omega)$ *and an* L^p*-viscosity supersolution* $\overline{u} \in C(\Omega)$ *of* (1) *such that*

$$\underline{u} \leq \overline{u} \quad \text{in } \Omega.$$

Then, setting $u(x) = \sup_{v \in \mathcal{S}} v(x)$ *for* $x \in \Omega$*, where*

$$\mathcal{S} := \left\{ v \in C(\Omega) \;\middle|\; \begin{array}{c} v \text{ is an } L^p\text{-viscosity subsolution of } (1) \\ \text{such that } \underline{u} \leq v \leq \overline{u} \text{ in } \Omega. \end{array} \right\},$$

we see that u *is an* L^p*-viscosity solution of* (1).

Remark. By virtue of Proposition 2, we only need to show that u is an L^p-viscosity supersolution of (1). To this end, following the argument in [8], we suppose that the conclusion fails. Then, we construct an L^p-viscosity subsolution $w \in \mathcal{S}$ such that $w(\hat{x}) > u(\hat{x})$ for certain $\hat{x} \in \Omega$. The only difference from the argument in [8] is to construct w so that it belongs to $C(\Omega)$. See [11] for the details.

4 Interior Hölder estimate

In this section, we consider the case when G is uniformly elliptic. To give the definition of uniform ellipticity, fixed $\lambda, \Lambda > 0$, we recall Pucci operators: For $X \in S^n$,

$$\mathcal{P}^+(X) = \max\{-\mathrm{Tr}(AX) \mid \lambda I \leq A \leq \Lambda I\} \quad \text{and} \quad \mathcal{P}^-(X) = -\mathcal{P}^+(-X).$$

In what follows, we suppose the following uniform ellipticity:

$$\left\{ \begin{array}{c} \mathcal{P}^-(X-Y) \leq G(x,r,q,X) - G(x,r,q,Y) \leq \mathcal{P}^+(X-Y) \\ \text{for } x \in \Omega, r \in \mathbf{R}, q \in \mathbf{R}^n, X, Y \in S^n. \end{array} \right. \tag{7}$$

To show the interior Hölder continuity estimate on L^p-viscosity solutions of (1) constructed via the above Perron's method, we need to modify Caffarelli's argument [2] (also [3]) for continuous viscosity solutions. In fact, we have to go back to the standard estimate for the oscillation of L^p-viscosity solutions. For the details, we refer to [11].

We do not know if our estimate is true for L^p-viscosity solutions in general because, there exists a gap to be fulfilled between the weak Harnack inequality and the local maximum principle.

In this section, for simplicity, we suppose that

$$G(x,0,O)=0 \quad \text{for } x\in\Omega. \tag{8}$$

Moreover, we suppose G to have quadratic growth with respect to Du:

$$\begin{cases} \text{For any compact set } K\subset\Omega, \exists L_K>0 \text{ such that} \\ \qquad |G(x,q,O)|\leq L_K(1+|q|^2) \\ \qquad\quad \text{for } x\in K, q\in\mathbf{R}^n. \end{cases} \tag{9}$$

Theorem 5. *Assume that* (5), (7), (8) *and* (9). *Let u be an L^p-viscosity solution of* (1) *via our Perron's method.*

Then, for each compact set $K\subset\Omega$, there is $\sigma=\sigma(K)\in(0,1)$ such that $u\in C^\sigma(K)$.

Remark. To deal with the quadratic nonlinearity assumption (9), we use two kinds of transformations for u. For the details, we refer to "A Beginner's Gide" [10].

5 An application

Following [1], we consider the PDE:

$$\alpha u-\frac{1}{2}\triangle u+\frac{1}{2}|Du|^2=f(x) \quad \text{in } \mathbf{R}^n, \tag{10}$$

where $\alpha>0$.

For simplicity, we suppose that

$$\begin{cases} (i) & f\in C(\mathbf{R}^n), \\ (ii) & \exists C_0>0 \text{ such that } 0\leq f(x)\leq C_0(1+|x|^2) \quad \text{in } \mathbf{R}^n. \end{cases} \tag{11}$$

In this section, we only give an existence result for (10) without using stochastic control.

Proposition 6. *Assume that* (11) *holds. Then, there exists a strong solution $u\in\cap_{p>1}W^{2,p}_{loc}(\mathbf{R}^n)$ of* (10).

Remark. For the uniqueness of strong solutions for (10), we need more hypotheses on f. See [1] for the details.

Sketch of proof. Notice that the PDE in (10) satisfies (5), (7), (8) and (9).

By (11), it is easy to show that $\underline{u} \equiv 0$ and $\overline{u} = \mu(1+|x|^2)$ are, respectively, a L^p-viscosity sub- and supersolutions for large $\mu > 1$. Thus, in view of Theorems 4 and 5, we can find an L^p-viscosity solution $u \in C(\mathbf{R}^n)$ of (10).

Hence, we easily observe that $v := e^{-u}$ is a viscosity solution of

$$-\frac{1}{2} \triangle v = v(\alpha - f) \quad \text{in } \mathbf{R}^n.$$

Thus, since $v \in W^{2,p}_{loc}(\mathbf{R}^n)$ for $p > 1$, by the local boundness of u, we obtain the same regularity for u. $\square$

6 Appendix

We only give a list of necessary propositions to prove Harnack inequality for "semicontinuous" L^p-viscosity solutions.

Throughout this section, we suppose that $\Omega \subset \mathbf{R}^n$ is a bounded domain.

For $v : \Omega \to \mathbf{R}$ and $r > 0$, we denote by $\Gamma_r[v, \Omega]$ the set

$$\Gamma_r[v, \Omega] = \{x \in \Omega \mid \exists p \in \overline{B}_r \text{ such that } v(y) \leq v(x) + \langle p, y - x \rangle \text{ for } y \in \Omega\}.$$

Proposition 7. (ABP maximum principle, cf. [4]) *Assume that* (2) *holds. There is* $C_0 := C_0(\Lambda/\lambda, n) > 0$ *such that if* $u : \overline{\Omega} \to \mathbf{R}$ *is a bounded L^p-viscosity subsolution of*

$$\mathcal{P}^-(D^2 u) = f \quad \text{in } \Omega,$$

and

$$r_0 := \max_{\overline{\Omega}} u^* - \max_{\partial\Omega} (u^*)^+ > 0$$

then

$$\max_{\overline{\Omega}} u^* \leq \max_{\partial\Omega} (u^*)^+ + C_0 \mathrm{diam}(\Omega)^{2-\frac{n}{p}} \|f^+\|_{L^p(\Gamma_{r_0/2}[u^*, \Omega])}.$$

Proposition 8. *There are* $p_0 = p_0(\Lambda/\lambda, n) > 0$ *and* $C_1 := C_1(\Lambda/\lambda, n) > 0$ *such that if* $u : B_{2\sqrt{n}} \to [0, \infty)$ *is a nonnegative L^p-viscosity supersolution of*

$$\mathcal{P}^+(D^2 u) = f \quad \text{in } B_{2\sqrt{n}},$$

then we have

$$\|u_*\|_{L^{p_0}(Q_1)} \leq C_1 \left(\inf_{Q_{1/2}} u_* + \|f^-\|_{L^p(B_{2\sqrt{n}})} \right).$$

Proposition 9. *For any $q > 0$, there is $C_2 := C_2(\Lambda/\lambda, n) > 0$ such that if for $f \in L^p(Q_2)$, $u : B_{2\sqrt{n}} \to [0, \infty)$ is an L^p-viscosity subsolution of*

$$\mathcal{P}^-(D^2 u) = f \quad \text{in } Q_2,$$

then we have

$$\sup_{Q_1} u^* \leq C_2 \left(\|(u^*)^+\|_{L^q(Q_2)} + \|f^+\|_{L^p(Q_2)} \right).$$

References

[1] O. ALVAREZ, A quasilinear elliptic equation in $\mathbf{R}^N$, *Proc. Roy. Soc. Edinburgh Sect. A*, **126** (5), (1996), 911-921.

[2] L. A. CAFFARELLI, Interior a priori estimates for solutions of fully nonlinear equations, *Ann. Math.*, **130** (1989), 189-213.

[3] L. A. CAFFARELLI & X. CABRÉ, Fully Nonlinear Elliptic Equations, Colloquium Publications **43**, AMS, 1995.

[4] L. A. CAFFARELLI, M. G. CRANDALL, M. KOCAN & A. ŚWIĘCH, On viscosity solutions of fully nonlinear equations with measurable ingredients, *Comm. Pure Appl. Math.* **49** (1996), 365-397.

[5] M.G. CRANDALL, H. ISHII, AND P.-L. LIONS, User's guide to viscosity solutions of second order partial differential equations, *Bull. Amer. Math. Soc.*, **27** (1992), 1-67.

[6] M. G. CRANDALL AND A. ŚWIĘCH, A note on generalized maximum principles for elliptic and parabolic PDE, *Lecture Notes in Pure and Appl. Math.*, **234** (2003), 121-127.

[7] D. GILBARG AND N. S. TRUDINGER, Elliptic Partial Differential Equations of Second Order, Second Edition, Springer-Verlag, 1983.

[8] H. ISHII, Perron's method for Hamilton-Jacobi equations, *Duke Math. J.*, **55** (1987), 369-384.

[9] H. ISHII AND P.-L. LIONS, Viscosity solutions of fully nonlinear second-order elliptic partial differential equations, *J. Differential Equations*, **83** (1990), 26-78.

[10] S. KOIKE, A Beginner's Guide to the Theory of Viscosity Solutions, *MSJ Memoirs*, **13**, Math. Soc. Japan, 2004.

[11] S. KOIKE, Perron's method and L^p-viscosity solutions, preprint.

[12] S. KOIKE AND A. ŚWIĘCH, Maximum principle and existence of L^p-viscosity solutions for fully nonlinear, uniformly elliptic equations with measurable and quadratic terms, to appear in *Nonl. Differential Equations Appl.*

[13] O. A. LADYZHENSKAYA AND N. N. URAL'TSEVA, Linear and Quasilinear Elliptic Equations, Academic Press, New York, 1968.

[14] P.-L. LIONS, A remark on Bony maximum principle, *Proc. Amer. Math. Soc.*, **88** (7), (1983), 503-508.

[15] N. NADIRASHVILI, Nonuniqueness in the martingale problem and the Dirichlet problem for uniformly elliptic operators, *Ann. Scoula Norm. Sup. Pisa Cl. Sci.*, **24** (4), (1997), 537-549.

[16] O. PERRON, Eine neue Behandlung der Randwertaufgabe für $\Delta u = 0$, *Math. Z.*, **18**, 42-54, (1923).

[17] M. V. SAFONOV, Nonuniqueness for second-order elliptic equations with measurable coefficients, *SIAM J. Math. Anal.*, **30** (4), (1999), 879-895.